Math 847 Qualifying Examination August 2022

Instructions: You must show all necessary work to get full or partial credits. You can use a 3×5 index card. You can not use your book, cell phone, computer, or other notes. Read all problems through once carefully before beginning work.

Notation: $\mathbb{R}^n$ denotes the standard Euclidean space with $|x| = \sqrt{x_1^2 + ... + x_n^2}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$. $\Delta u(x) = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j \partial x_j}$ stands for the Laplace operator in $\mathbb{R}^n$. Ω is used for any open, bounded, and smooth domain in $\mathbb{R}^n$ with $\partial\Omega$ as its boundary, and $\nu(x)$ is the unit out normal at $x \in \partial\Omega$. ω_n is the surface area for $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.

Problem 1 *Consider the first order equation* $u_t(x,t) + 2tu_x(x,t) = u$, $x \in \mathbb{R}$, $t \in \mathbb{R}$.

(a) Find a solution $u(x,t)$ *with initial data* $u(x,0) = \sin x$, $x \in \mathbb{R}$.

(b) What is the characteristic curve starting from $x_0 = 0, t_0 = 0$ *? Can you find a* C^1 *solution in a neighborhood of* $(0,0)$ *such that* $u(s^2, s) = \cos s$*? Explain your answer.*

Problem 2 *Let* $\Omega = \{(x,y) \in R^2 \mid \|(x,y)\| = \sqrt{x^2+y^2} < 1\}$. *Assume that* u *is a smooth function on* $\overline{\Omega}$ *such that*

$$\begin{cases} \Delta u = 0, & (x,y) \in \Omega, \\ u(x,y) = 2x^2 - y^2 + 6y, & x^2 + y^2 = 1. \end{cases}$$

(a) What is $u(0,0)$*? What theorem did you use?*

(b) What is the maximum and minimum values of u *on* $\overline{\Omega}$*? What theorem did you use?*

(c) Find a point (x_0, y_0) *on* $\partial\Omega$ *such that* $\frac{\partial u}{\partial \nu}(x_0, y_0) > 0$*? Briefly explain your answer, where* ν *is the unit outnormal at the point* (x_0, y_0).

Problem 3 *Let $g(x)$ be a continuous function on $\mathbb{R}^n$, $0 < g(x) \le 1$ for all $x \in \mathbb{R}^n$, and $\int_{\mathbb{R}^n} g(x)dx = 1$. Consider the initial value problem*

$$\begin{cases} u_t(x,t) = \triangle u(x,t) - u, & x \in R^n,\ t > 0, \\ u(x,0) = g(x), & x \in R^n. \end{cases}$$

(a) Prove that there is a solution $u(x,t)$ such that $0 < u(x,t) < e^{-t}\min\{1, \frac{1}{(4\pi t)^{n/2}}\}$ for all $x \in \mathbb{R}^n$ and $t > 0$.

(b) Can you find another different solution to the initial value problem? Briefly explain your answer.

(c) Can you find another different positive solution $u(x,t)$ to the initial value problem? Briefly explain your answer.

(d) For any continuous function $f(x,t)$ such that

$$|f(x,t)| \le 5, \quad x \in \mathbb{R}^n,\ t \ge 0,$$

prove that there is a bounded solution to the following non-homogeneous equation

$$\begin{cases} u_t = \triangle u - u + f(x,t), & x \in R^n,\ t > 0, \\ u(x,0) = g(x), & x \in R^n. \end{cases}$$

Problem 4 *Let $u(x,y,z,t)$ be a solution of the initial value problem for the wave equation in $R^3 \times R$*

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + u_{zz}, & (x,y,z) \in \mathbb{R}^3,\ t \in \mathbb{R} \\ u(x,y,z,0) = 0, & (x,y,z) \in \mathbb{R}^3 \\ u_t(x,y,z,0) = \frac{1}{(4+x^2+y^2+z^2)^2} & (x,y,z) \in \mathbb{R}^3. \end{cases}$$

(a) Find an explicit formula for $u(0,0,0,t)$. Briefly explain your answer.

(b) Find a solution of the form $u(x,y,z,t) = V(r,t)$ with $r = \sqrt{x^2+y^2+z^2}$.

[Hint: Note that $W(r,t) = rV(r,t)$ satisfies $W_{rr} = W_{tt}$).]

(c) Check that $V(0,t) = u(0,0,0,t)$ you obtained from Parts (a) and (b).

(d) Is it possible that $u(x,t)$ such that $u(5,0,0,2) - u(0,3,4,2) = 2$? Explain your answer.

(e) Prove that there is a constant $A > 0$ such that

$$F_1(t) = \int_{\mathbb{R}^3} u(x,t)dx = A\,t.$$

Problem 5 *Consider the second order differential equation*

$$\Delta u(x) = f(x_1, x_2, ..., x_n) = \frac{3 + \cos x_1 + \sin(x_1 + x_2 + ...x_n)}{(1+|x|^2)^n}, \quad x \in \mathbb{R}^n. \qquad (1)$$

Note that $\frac{1}{(1+|x|^2)^n} \le f(x) \le \frac{5}{(1+|x|^2)^n}$.

(a) Prove that there are infinitely many solutions to this equation with $u(0) = 1$ when $n \ge 3$.

(b) Prove that there is one and only one solution $u(x)$ such that $u(x)$ is bounded and $u(0) = 1$ when $n \ge 3$.

(c) Let $g(r) = \int_{|y|=1} u(ry)dS_y$. Find $g'(r)$ in terms of the integral of $f(x)$

(d) Prove that $g(r)$ is an increasing and bounded function for any solution $u(x)$ of (1) when $n \ge 3$ (even for unbounded solutions of (1) .

(e) For $n = 2$, prove that there is no bounded solution to (1) by showing that $\lim_{r\to\infty} g(r) = \infty$.

Problem 6 *Consider the nonlinear heat equation*

$$\begin{cases} u_t = u_{xx} + u_{yy} - |Du|^2, & x^2+y^2 < 1, t > 0, \\ u(x,y,0) = (1-x^2-y^2)^2, & x^2+y^2 < 1, \\ u(x,y,t) + xu_x(x,y,t) + yu_y(x,y,t) = u + Du \cdot \nu = 0, & x^2+y^2 = 1, t \ge 0. \end{cases}$$

(a) If $u(x,t)$ is a smooth solution of the equation, prove that u can not attain its positive maximum or negative minimum on the lateral boundary $x^2+y^2 = 1$.

(b) If $u(x,t)$ is a smooth solution of the equation, prove that

$$0 \le u(x,y,t) \le 1 \textit{ for all } x^2+y^2 < 1 \textit{ and } t > 0 \ .$$

(c) Prove that the equation has at most one solution.